第09章 Declaring Type and Type Class

◇ Type Declaration

In Haskell, a new name for an existing type can be defined using a **type declaration**.

```
type String = [Char]
```

- **String** is a synonym for the type **[Char]**

Type declarations can be used to make other types easier to read.

For example, given:

```
type Pos = (Int, Int)
```

we can define:

```
origin :: Pos
origin  =  (0, 0)

left :: Pos -> Pos
left (x, y)  =  (x - 1, y)
```

Like function definitions, type declarations can also have parameters.

For example, given:

```
type Pair a = (a, a)
```

we can define:

```
mult :: Pair Int -> Int
mult (m, n)  =  m * n

copy :: a -> Pair a
copy x  =  (x, x)
```

Type declarations can be nested:

```
type Pos = (Int, Int)
type Trans = Pos → Pos
```

However, they cannot be recursive:

```
type Tree = (Int,[Tree])
```

- error: Cycle in type synonym declarations

◇ Data Declaration

A completely new type can be defined by specifying its values using a data declaration.

```
data Bool = False | True
```

- Bool is a new **type**, with two new **values** False and True.
- Bool is a **type constructor**, and False/True is a **data constructor**.

  Type/Data constructor names must always begin with an upper-case letter.
- Data declarations are similar to context free grammars.

  The former specifies the values of a type, the latter the sentences of a language.

Values of new types can be used in the same ways as those of built in types.

For example, given:

```
data Answer  =  Yes | No | Unknown
```

we can define:

```
answers :: [Answer]
answers  =  [Yes, No, Unknown]

flip :: Answer → Answer
flip Yes  =  No
flip No  =  Yes
flip Unknown  =  Unknown
```

The data constructors can also have parameters.

For example, given:

```
data Shape = Circle Float | Rect Float Float
```

we can define:

```
square :: Float -> Shape
square n  =  Rect n n

area :: Shape -> Float
area (Circle r)  =  pi * r^2
area (Rect x y)  =  x * y
```

```
data Shape = Circle Float | Rect Float Float
```

- `Shape` has values of the form `Circle r` where r is a Float, and `Rect x y` where x and y are Float.
- `Circle` and `Rect` can be viewed as functions that construct values of type Shape:

  ```
  Circle :: Float -> Shape
  Rect :: Float -> Float -> Shape
  ```

The type constructors can also have parameters.

For example, given:

```
data Maybe a = Nothing | Just a
```

we can define:

```
safediv :: Int -> Int -> Maybe Int
safediv _ 0 = Nothing
safediv m n = Just $ div m n

safehead :: [a] -> Maybe a
safehead [] = Nothing
safehead xs = Just $ head xs
```

◇ Recursive Type

In Haskell, new types can be declared in terms of themselves.

That is, types can be recursive.

```
data Nat = Zero | Succ Nat
```

- Nat is a new type, with two data constructors

  Zero :: Nat

  Succ :: Nat → Nat

- A value of type Nat is

  either Zero, or of the form Succ n where n :: Nat.

- That is, Nat contains the following infinite sequence of values:

  Zero, Succ Zero, Succ $ Succ Zero, Succ $ Succ $ Succ Zero, ...

```
data Nat = Zero | Succ Nat
```

- We can think of values of type Nat as natural numbers, where:
  - Zero represents 0, and
  - Succ represents the function (1+).
- For example, the value

  Succ $ Succ $ Succ Zero

  represents the natural number

  (1+) $ (1+) $ (1+) 0

```
data Nat = Zero | Succ Nat
```

- Using recursion, it is easy to define functions that convert between values of type Nat and Int:

```
nat2int :: Nat → Int
nat2int Zero     = 0
nat2int (Succ n) = 1 + nat2int n

int2nat :: Int → Nat
int2nat 0 = Zero
int2nat n = Succ $ int2nat $ n - 1
```

```
data Nat = Zero | Succ Nat
```

- Two naturals can be added by converting them to integers, adding, and then converting back:

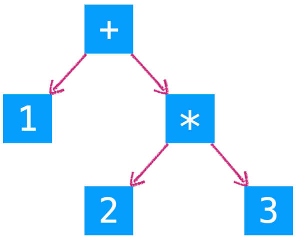
  ```
  add :: Nat → Nat → Nat
  add m n = int2nat $ nat2int m + nat2int n
  ```

  However, using recursion the function add can be defined without the need for conversions:

  ```
  add Zero     n = n
  add (Succ m) n = Succ $ add m n
  ```

◇ **Example**: A Type for Arithmetic Expressions

Consider a simple form of expressions built up from integers using **addition** and **multiplication**.



Can we define a type to represent this kinds of arithmetic expressions ?

Using recursion, a suitable new type to represent such expressions can be declared by:

```
data Expr = Val Int | Add Expr Expr | Mul Expr Expr
```

- 1 + (2 * 3)   ===>   Add (Val 1) (Mul (Val 2) (Val 3))

Using recursion, it is now easy to define functions that process expressions. For example:

```
data Expr = Val Int | Add Expr Expr | Mul Expr Expr

size :: Expr -> Int
size (Val n)   = 1
size (Add x y) = size x + size y
size (Mul x y) = size x + size y

eval :: Expr -> Int
eval (Val n)   = n
eval (Add x y) = eval x + eval y
eval (Mul x y) = eval x * eval y
```

```
data Expr = Val Int | Add Expr Expr | Mul Expr Expr
```

- `The three constructors have types:`

  ```
  Val :: Int -> Expr
  Add :: Expr -> Expr -> Expr
  Mul :: Expr -> Expr -> Expr
  ```

对于类型 `Expr`，是否存在一个对应的`fold`函数呢？

如果你真正理解了`Natural`和`List`上的`fold`函数，这就是一件非常简单的事情。

- 把这三个`data constructors`替换为恰当的三个函数

不妨将`Expr`上的`fold`函数命名为 `folde`。可知，其类型如下：

```
folde :: (Int -> a) -> (a -> a -> a) -> (a -> a -> a) -> Expr -> a
-- 如何定义这个函数，是本章的一个作业题
```

基于`folde`，可以对刚才定义的两个函数 `size` 和 `eval` 进行重新定义：

```
size :: Expr -> Int
size (Val n)   = 1
size (Add x y) = size x + size y
size (Mul x y) = size x + size y
```

```
size :: Expr -> Int
size = folde (\x -> 1) (+) (+)
```

```
eval :: Expr -> Int
eval (Val n)   = n
eval (Add x y) = eval x + eval y
eval (Mul x y) = eval x * eval y
```

```
eval :: Expr -> Int
eval = folde id (+) (*)
```

◇ Newtype Declaration

If a new type has a single constructor with a single argument, then it can also be declared using the **newtype** mechanism.

For example:

```
newtype Nat = N Int
```

另外两种声明方式:

- ○ `data Nat = N Int` less efficient
- ○ `type Nat = Int` less safe

◇ Type class and instance declaration

Declare a type class:

```
class Eq a where
    (==), (/=) :: a -> a -> Bool
    x /= y = not (x == y)
    x == y = not (x /= y)
    {-# MINIMAL (==) | (/=) #-}
```

- For a type a to be an instance of the class Eq,
  it must support equality and inequality operators of the

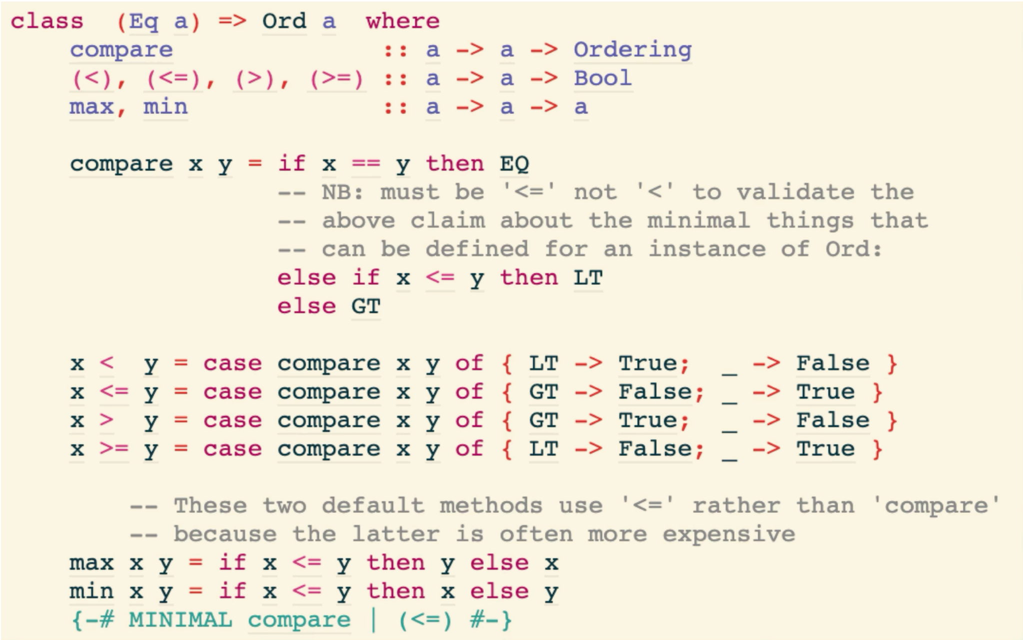
specified types

Declare that a type is an instance of a type class:

```
instance Eq Bool where
  False == False  =  True
  True == True  =  True
  _ == _  =  False
```

- Only types that are declared using the **data** and **newtype** mechanisms can be made into instances of type classes.
- Default definitions can be overridden in instance declarations if desired.

Type classes can also be extended to form new type classes.



```
class  (Eq a) => Ord a  where
    compare              :: a -> a -> Ordering
    (<), (<=), (>), (>=) :: a -> a -> Bool
    max, min             :: a -> a -> a

    compare x y = if x == y then EQ
                  -- NB: must be '<=' not '<' to validate the
                  -- above claim about the minimal things that
                  -- can be defined for an instance of Ord:
                  else if x <= y then LT
                  else GT

    x <  y = case compare x y of { LT -> True;  _ -> False }
    x <= y = case compare x y of { GT -> False; _ -> True }
    x >  y = case compare x y of { GT -> True;  _ -> False }
    x >= y = case compare x y of { LT -> False; _ -> True }

        -- These two default methods use '<=' rather than 'compare'
        -- because the latter is often more expensive
    max x y = if x <= y then y else x
    min x y = if x <= y then x else y
    {-# MINIMAL compare | (<=) #-}
```

```
instance Ord Bool where
   False <= _  =  True
   True <= True  =  True
   _ <= _  =  False
```

◇ Derived instances

When new types are declared, it is usually appropriate to make them into instances of a number of built-in classes.

```
data Bool = False | True  deriving  (Eq, Ord, Show, Read)
```

```
ghci> False < True
True
ghci> False == True
False
```

◇ **Example**: Tautology Checker / 重言检查器

The Problem: Develop a function that decides if a simple **propositional formula** is always true.

```
1. A ∧ ¬A
2. (A ∧ B) ⇒ A
3. A ⇒ (A ∧ B)
4. (A ∧ (A ⇒ B)) ⇒ B
```

求解方法：查看命题公式的真值表，判断是否所有结果都为**真**。

A	B	A ∧ B
F	F	F
F	T	F
T	F	F
T	T	T

A	B	A ⇒ B
F	F	T
F	T	T
T	F	F
T	T	T

A	¬A	A ∧ ¬A
F	T	F
T	F	F

A	B	(A ∧ B) ⇒ A
F	F	T
F	T	T
T	F	T
T	T	T

A	B	A ⇒ (A ∧ B)
F	F	T
F	T	T

A	B	(A ∧ (A ⇒ B)) ⇒ B
F	F	T
F	T	T

T	*F*	*F*
T	*T*	*T*

T	*F*	*T*
T	*T*	*T*

第1步：定义一个用于表示命题公式的类型

```haskell
data Prop = Const Bool
            | Var   Char
            | Not   Prop
            | And   Prop Prop
            | Imply Prop Prop


-- 1.  A ∧ ¬A
p1 = And (Var 'A') (Not (Var 'A'))

-- 2.  (A ∧ B) => A
p2 = Imply (And (Var 'A') (Var 'B')) (Var 'A')

-- 3.  A => (A ∧ B)
p3 = Imply (Var 'A') (And (Var 'A') (Var 'B'))

-- 4.  (A ∧ (A => B)) => B
p4 = Imply (And (Var 'A') (Imply (Var 'A') (Var 'B'))) (Var 'B')
```

第2步：定义函数 `vars :: Prop -> [Char]`，求出一个命题公式中的变量

```haskell
vars :: Prop -> [Char]
vars (Const _)    = []
vars (Var x)      = [x]
vars (Not p)      = vars p
vars (And p q)    = vars p ++ vars q
vars (Imply p q) = vars p ++ vars q
```

```
ghci> var p4
"AABB"
```

第3步：定义一个类型，用于表达命题变量与真/假值之间的绑定/置换关系

```
type Subst = Assoc Char Bool
type Assoc k v = [(k, v)]

-- example
subst :: Subset
subst = [ ('A' ,True),  ('B', False) ]
```

第4步：定义函数 `bools :: Int -> [[Bool]]`，用于生成n个bool类型值所有可能的排列

```
bools :: Int -> [[Bool]]
bools 0 = [[]]
bools n = map (False :) bss ++ map (True :) bss
    where bss = bools $ n - 1
```

```
ghci> bools 2
[[False,False],[False,True],[True,False],[True,True]]
```

第5步：定义函数 `varSubsts :: [Char] -> [Subst]`：接收一组命题变量，生成对这些变量所有可能的赋值/置换方式

```
varSubsts :: [Char] -> [Subst]
varSubsts vs = map (zip vs) (bools $ length vs)
```

```
 ghci> varSubsts "AB"
 [ [('A',False),('B',False)],
   [('A',False),('B',True)],
   [('A',True),('B',False)],
   [('A',True),('B',True)] ]
```

第6步：定义函数 `eval :: Subst -> Prop -> Bool`：给定一个命题公式和一个置换表，评估这个命题公式的值

```
eval  :: Subst → Prop → Bool
eval  _ (Const b)  = b
eval s (Var x)      = find x s
eval s (Not p)      = not (eval s p)
eval s (And p q)   = eval s p  &&  eval s q
eval s (Imply p q) = eval s p  ⩽  eval s q
--                                 ^^ 注意：这里出现了一件很有趣的事情
```

第7步：定义函数 `isTaut :: Prop → Bool`：判断一个命题公式是否重言

```
isTaut :: Prop → Bool
isTaut p = and [eval s p | s ← varSubsts vs]
     where  vs = rmdups (vars p)
```

```
ghci> isTaut p1
False
ghci> isTaut p2
True
ghci> isTaut p3
False
ghci> isTaut p4
True
```

◇ **Example**: Abstract Machine

我们可以定义一个表示“算术运算表达式”的类型，然后定一个评估函数，对一个表达式进行求值：

```
data Expr = Val Int | Add Expr Expr

value :: Expr → Int
value (Val n)     = n
value (Add x y)  = value x + value y
```

例如，对于表达式 `(2 + 3) + 4`，其求值过程如下：

```
     value (Add (Add (Val 2) (Val 3)) (Val 4))
===     { applying value }
```

```
    value (Add (Val 2) (Val 3))  +  value (Val 4)
≡     { applying the first value }
    (value (Val 2)  +  value (Val 3))  +  value (Val 4)
≡     { applying the first value}
    (2  +  value (Val 3))  +  value (Val 4)
≡     { applying the first value}
    (2  +  3)  +  value (Val 4)
≡     { applying the first + }
    5  +  value (Val 4)
≡     { applying value }
    5  +  4
≡     { applying + }
    9
```

注意：

- ◇ 在类型声明中，我们并未指定表达式求值的详细步骤
- ◇ Haskell语言的编译器在背后帮我们做了很多的事情。

一个问题：可以自定义表达式的求值步骤吗？

下面是一个解决方案：

```
data Expr  =  Val Int | Add Expr Expr

value :: Expr -> Int
value e  =  eval e []

type Cont  =  [Op]
data Op  =  EVAL Expr | ADD Int

eval :: Expr -> Cont -> Int
eval (Val n) c  =  exec c n
eval (Add x y) c  =  eval x $ EVAL y : c

exec :: Cont -> Int -> Int
exec [] n  =  n
exec (EVAL y : c) n  =  eval y $ ADD n : c
exec (ADD n : c) m  =  exec c $ n + m
```

作业01

Using recursion and the function add, define a function that multiplies two natural numbers.

作业02

Define a suitable function **folde** for expressions and give a few examples of its use.

作业03

Define a type **Tree a** of binary trees built from **Leaf** values of type a using a **Node** constructor that takes two binary trees as parameters.